الجامعة الأردنية	نظرية أعداد: الامتحان الثاني	الأحد ٢٠١٦/٧/٢٤
اسم الطالب: ---		الرقم الجامعي: ------------------

[1] (3 marks) Show that if $c|a$ and $c|b$, then $c|\gcd(a,b)$.

[2] (5 marks) Determine all positive solutions of the Diophantine equation

$54x + 21y = 906$

$54 = (21)(2) + 12$

$21 = (12)(1) + 9$

$12 = (9)(1) + 3$

$9 = (3)(3) + 0$

$\gcd(54, 21) = 3 \mid 906$

$3 = 12 - 9$

$= 12 - (21 - 12)$

$= (12)(2) - 21$

$= (54 - (21)(2))2 - 21$

$3 = (54)(2) + (21)(-5)$

$906 = (54)(604) + (21)(-1510)$

$x = 604 + 7t$

$y = -1510 - 18t \qquad t \in \mathbb{Z}$

$604 + 7t > 0 \Rightarrow t > -\frac{604}{7} \approx -86.2$

$-1510 - 18t > 0 \Rightarrow t < -\frac{1510}{18} \approx -83.8$

$-86.2 < t < -83.8$

[3] (3 marks) Show that if $n > 1$, then n is prime or has a prime divisor.

[4] (3 marks) Assume $\gcd(a, b) = 1$. Prove that $\gcd(a + b, a^2 - ab + b^2) = 1$ *or* 3

(Hint: $a^2 - ab + b^2 = (a + b)^2 - 3ab$)

Let $d = \gcd(a+b, a^2-ab+b^2)$

$a+b = dx$

$a^2-ab+b^2 = dy \quad x,y \in \mathbb{Z}$.

$(a+b)^2 - 3ab = dy$

$3ab = d^2x^2 - dy = d(dx^2 - y)$

$3a^2 + 3ab = 3adx$

$d \mid 3a^2$

$3ab + 3b^2 = 3bdx$

so $d \mid 3b^2$

$d \mid \gcd(3a^2, 3b^2) = 3\gcd(a^2, b^2) = 3$

$d = 1 \vee d = 3$

[5] (3 marks) Prove that if $a|(bc)$ and $\gcd(a, b) = 1$, the $a|c$.

[6] (3 marks) Show that there are infinitely many primes.

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[1] (3 marks) Show that if $c|a$ and $c|b$, then $c|\gcd(a,b)$.

$\gcd(a,b) = au + bv \quad u, v \in \mathbb{Z}$

$c|a \wedge c|b$ implies $c|ax+by \quad \forall x, y \in \mathbb{Z}$

So $c | au+bv = \gcd(a,b)$

[2] (5 marks) Determine all positive solutions of the Diophantine equation

$123x + 360y = 99$

$360 = (123)(2) + 114$

$123 = (114)(1) + 9$

$114 = (9)(12) + 6$

$9 = (6)(1) + 3$

$6 = (3)(2) + 0$

$\gcd(123, 360) = 3 \,|\, 99$

$3 = 9 - 6$

$= 9 - (114 - (9)(12))$

$= (9)(13) - 114$

$= (123 - 114)(13) - 114$

$= (123)(13) + (-14)(114)$

$= (123)(13) + (-14)(360 - (123)(2))$

$3 = (123)(41) + 360(-14)$

$99 = (123)(1353) + (360)(-462)$

$x = 1353 + 120t$

$y = -462 - 41t$

$t > -\frac{1353}{120} = -11.275$

$t < -\frac{462}{41} \approx -11.268$

There is no +ve solution

[3] (3 marks) Show that if $n > 1$, then n is prime or has a prime divisor.

[4] (3 marks) Assume $\gcd(a, b) = 1$. Prove that $\gcd(a + b, a^2 + b^2) = 1$ *or* 2

(Hint: $a^2 + b^2 = (a + b)(a - b) + 2b^2$))

Let $d = gcd(a+b, a^2+b^2)$

$a+b = dx$

$a^2+b^2 = dy \qquad x, y \in \mathbb{Z}$

$(a+b)(a-b) + 2b^2 = dy$

$2b^2 = dy - dx(a-b) = d(y - x(a-b))$

$(a+b)(b-a) + 2a^2 = dy$

$2a^2 = dy - dx(b-a) = d(y - x(b-a))$

So,

$d \mid gcd(2b^2, 2a^2)$

$= 2gcd(b^2, a^2)$

$= (2)(1)$

So $d=1 \vee d=2$

[5] (3 marks) Prove that if $a|c$, $b|c$ and $\gcd(a, b) = 1$, the $ab|c$.

[6] (3 marks) Show that there are infinitely many primes.